INVESTIGATION OF CHARGED GAS INSTABILITY IN TWO–DIMENSIONAL MODEL OF LIGHTNING DISCHARGE

O. V. KRAVCHENKO[1]*, V. I. PUSTOVOIT[2]

[1] Bauman Moscow State Technical University, Moscow, 105005, Russia
[2] Scientific and Technological Center of Unique Instrumentation RAS, Moscow, 117342, Russia
*olekravchenko@gmail.com

ABSTRACT

The present paper is an extension of previous research related to the problem of the leader of lightning formation. It was introduced earlier that a homogenous charged media could become unstable under some initial conditions in [1], [4] in one dimensional case. We present here an investigation of such kind instability in two dimension case also.

1. INTRODUCTION

Since the middle of the last century problems of stability of charged jet flow occupy the central place in EHD. But, the fluid equations are difficult to handle because of their nonlinearity and mathematical complexity as a result.

One of the approaches is to study the influence of an electric field on a velocity field. E. Moreau and O.Vallee considered in [2] an electric field as an elastic term of force for some model plasma problem. An analytic resolution of a one dimensional problem faced in ionised media was proposed, when the electric field is dominant in front of the magnetic field (electro hydrodynamical approach).

This situation could be appeared for example in electric discharges or electric arcs studies. For this purpose, E. Moreau and O.Vallee considered a constant electric field applied to a plasma in which it was assumed the existence of a constant electron flow (created for example by electrodes system and depending of applied electric field), which may be interpreted as a source term. Moreover, the plasma was assumed to be composed of electrons and motionless ions.

After some reasoning a one dimensional Burgers like equation of the local electric field E_{loc} was appeared with an elastic forcing term

$$\partial_t E_{loc} = \nu \partial_{xx} E_{loc} + E_{loc} \partial_x E_{loc} - xS + c(t).$$

Finally the possibility to derive a new equation of evolution for the electric field held to the Poisson equation was shown.

It seems interesting that probably the first appearance of the Burgers like equation in plasma model of instability saturation by resonant mode coupling was considered by E. Ott, W. M. Manheimer, D. L. Book and J. P. Boris in [3].

Their model was in general form:

$$\partial_t u + \alpha_1 u \partial_x u + M(u) = 0,$$

where $M(u)$ is a linear operator given by

$$\int_{-\infty}^{\infty} M(u) e^{-ikx}\, dx = i\omega(k) \int_{-\infty}^{\infty} u e^{-ikx}\, dx,$$

where $\omega(k)$ is a dispersion relationship. It was noticed that in case

$$\omega(k) = i(\alpha_3 - k^2 \alpha_2),$$

a Burgers like equation was obtained in the following form

$$\partial_t u + \alpha_1 u \partial_x u - \alpha_2 \partial_{xx} u - \alpha_3 u = 0.$$

A variety of different instabilities that can be stabilized nonlinearly by resonant mode coupling were examined. A steady state behaviour of Burgers like equation was examined by a phase plane analysis also.

Finally, a simple electro hydrodynamical model describing lightning was proposed in [1]. A general self—similar solution to the nonlinear equation was obtained in one dimensional case. It was demonstrated that the uniform electric field and charge density distributions in a cloud could become unstable. A virtually periodic structure of the field and charge densities is formed by spatial redistribution.

Afterwards a numerical solution of Burgers like equation by wavelet Haar method was considered in static case [4].

The extension in two dimensional case of the model of such kind instability [1] is derived in section 2.

2. GOVERNMENT EQUATIONS

Let's consider a two–component fluid model of electron and ion fluids. The government equations in vector form are the following:

$$en_e\mathbf{E} + kT\nabla n_e + mn_e\nu\mathbf{v} = \mathbf{0}, \tag{1}$$
$$\partial_t n_e + \nabla(n_e\mathbf{v}) = 0, \tag{2}$$
$$\varepsilon_0\nabla\cdot\mathbf{E} = -4\pi(n_e - n_i). \tag{3}$$

The first equation (1) is a balance equation, the second one (2) is a continuity and the third (3) is a Gauss law. Under the following dimensionless variables

$$z = \frac{n_e}{n_i},\ \mathbf{u} = \frac{\mathbf{v}_e}{v_\star},\ l = \frac{2kT}{eE_\star},\ \mathbf{y} = \frac{\mathbf{E}}{E_\star},$$
$$\xi = \frac{x}{l}, \eta = \frac{y}{l}, v_\star = \frac{eE_\star}{m\nu}, \beta^2 = \frac{2P_0}{D_\star},$$
$$\tau = \frac{t}{t_\star},\ t_\star = \frac{l}{v_\star}, \tag{4}$$

system of Eqns.(1)–(3) transform into

$$\nabla z + (\mathbf{y}+\mathbf{u})z = \mathbf{0}, \tag{5}$$
$$\partial_\tau z + \nabla\cdot(z\mathbf{u}) = 0, \tag{6}$$
$$\nabla\cdot\mathbf{y} = 0.5\beta^2(1-z). \tag{7}$$

There are two different ways of reduction of (5)–(7) The dimensionless system of equations (5)–(7) could be reduced to vector equation (14) which , or it could be transformed into system (18), (19). From Eq.(3) obtain:

$$z = 1 - (2/\beta^2)\nabla\cdot\mathbf{y}, \tag{8}$$
$$\nabla z = -(2/\beta^2)\nabla(\nabla\cdot\mathbf{y}), \tag{9}$$
$$\partial_\tau z = -(2/\beta^2)\partial_\tau(\nabla\cdot\mathbf{y}). \tag{10}$$

From Eq.(1) obtain:

$$z\mathbf{u} = -z\mathbf{y} - \nabla z, \tag{11}$$
$$\nabla\cdot(z\mathbf{u}) = -\nabla\cdot(z\mathbf{y}) - \nabla^2 z. \tag{12}$$

Substituting the Eqns.(8)–(12) into (6) obtain the following:

$$(-2/\beta^2)\partial_\tau(\nabla\cdot\mathbf{y}) - \nabla\cdot(z\mathbf{y}+\nabla z) = 0,$$
$$\nabla\cdot\big((2/\beta^2)\partial_\tau\mathbf{y} + z\mathbf{y} + \nabla z\big) = 0,$$
$$\nabla\cdot\big(\partial_\tau\mathbf{y} + (0.5\beta^2 - \nabla\cdot\mathbf{y})\mathbf{y} - \nabla(\nabla\cdot\mathbf{y})\big) = 0. \tag{13}$$

Denote

$$\mathbf{F} := \partial_\tau\mathbf{y} + (0.5\beta^2 - \nabla\cdot\mathbf{y})\mathbf{y} - \nabla(\nabla\cdot\mathbf{y}),$$

then rewrite (13) as

$$\nabla\cdot\mathbf{F} = 0. \tag{14}$$

Obviously equation (14) is a divergent equation. So, there could be a set of cases.

Case A (partial): $\mathbf{F} = -\mathbf{g}$ is partial and Eq. (14) becomes follows in vector form

$$\partial_\tau\mathbf{y} = \nabla(\nabla\cdot\mathbf{y}) + \mathbf{y}(\nabla\cdot\mathbf{y}) - 0.5\beta^2\mathbf{y} + \mathbf{g}, \tag{15}$$

where source $\mathbf{g} = (g_1, g_2)$ depends on τ, so $\mathbf{g} = \mathbf{g}(\tau)$.

According to vector operators in cartesian:

$$\nabla\cdot\mathbf{y} = \partial_\xi y_1 + \partial_\eta y_2,$$
$$\nabla(\nabla\cdot\mathbf{y}) = (\partial_{\xi\xi}y_1 + \partial_{\xi\eta}y_2; \partial_{\xi\eta}y_1 + \partial_{\eta\eta}y_2),$$
$$\mathbf{y}(\nabla\cdot\mathbf{y}) = (y_1\partial_\xi y_1;\ y_2\partial_\eta y_2),$$
$$\partial_\tau\mathbf{y} = (\partial_\tau y_1;\ \partial_\tau y_2),$$

obtain from (15) a system of two scalar coupled Burgers like equations with linear $-0.5\beta^2\mathbf{y}$ and source term $\big(\mathbf{g} = (g_1; g_2)\big)$:

$$\begin{cases}\partial_\tau y_1 = \partial_{\xi\xi}y_1 + \partial_{\xi\eta}y_2 + y_1(\partial_\xi y_1 - 0.5\beta^2) + g_1,\\ \partial_\tau y_2 = \partial_{\eta\eta}y_2 + \partial_{\xi\eta}y_1 + y_2(\partial_\eta y_2 - 0.5\beta^2) + g_2.\end{cases} \tag{16}$$

Case B (general): $\nabla\cdot\mathbf{F} = 0$ is general. Define scalar function $s = s(\xi,\eta)$ as $s = \nabla\cdot\mathbf{y}$, then

$$\nabla\cdot\mathbf{F} \equiv \partial_\tau s - \nabla^2 s - s^2 - \mathbf{y}\cdot(\nabla s) + 0.5\beta^2 s = 0,$$

and equation (14) transform into system of two scalar equations

$$\begin{cases} \partial_\tau s = \nabla^2 s + s^2 + y_1 \partial_\xi s + y_2 \partial_\eta s - 0.5\beta^2 s, \\ \partial_\xi y_1 + \partial_\eta y_2 = s. \end{cases} \tag{17}$$

Let's obtain from (5)

$$\begin{aligned} z\mathbf{u} &= -z\mathbf{y} - \nabla z, \\ \nabla \cdot (z\mathbf{u}) &= -\nabla(z\mathbf{y}) - \nabla^2 z, \\ \nabla \cdot (z\mathbf{u}) &= -z(\nabla \cdot \mathbf{y}) - \mathbf{y} \cdot (\nabla z) - \nabla^2 z, \end{aligned}$$

and substitute in (6)

$$\partial_\tau z = \nabla^2 z + \mathbf{y} \cdot (\nabla z) + 0.5\beta^2 z(1 - z), \tag{18}$$

$$\nabla \cdot \mathbf{y} = 0.5\beta^2 (1 - z). \tag{19}$$

3. NUMERICAL SIMULATION

The system of coupled equations (16) could be solved numerically in square

$$(\xi,\ \eta) \in [0,\ 10] \times [0,\ 10]$$

under some conditions.

If we define source functions as follows

$$g_1(x, y) = -0.01x, \quad g_2(x, y) = -0.01y,$$

set parameter β and final time

$$\beta = \sqrt{2}, \quad t_{fin} = 10,$$

then the initial distributions of $y_1(\xi, \eta, 0)$, $y_2(\xi, \eta, 0)$ will be like in Fig. 1–3.

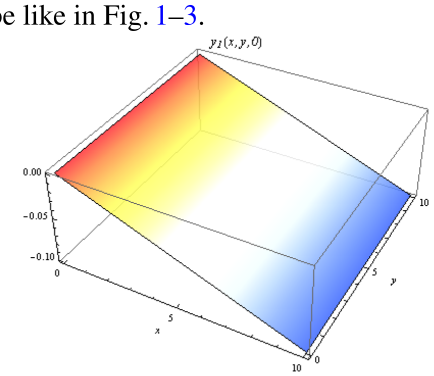

Fig. 1: Initial distribution of $y_1(\xi, \eta, 0)$

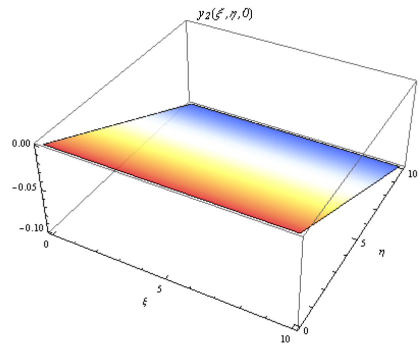

Fig. 2: Initial distribution of $y_2(\xi, \eta, 0)$

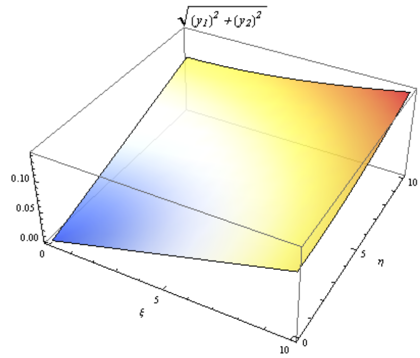

Fig. 3: Initial distribution of $|\mathbf{y}|$ *at* $t = 0$

Afterwards, the final distributions of the dimensionless $\mathbf{y}$ components $y_1(\xi, \eta, t_{fin})$, $y_2(\xi, \eta, t_{fin})$ are shown in Fig. 4–6.

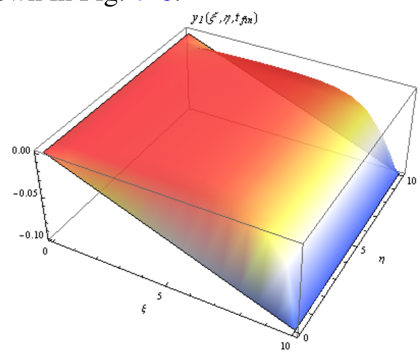

Fig. 4: Final distribution of $y_1(\xi, \eta, t_{fin})$

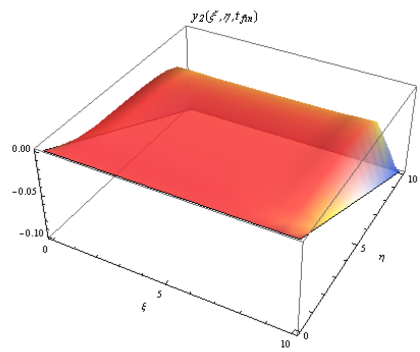

Fig. 5: Final distribution of $y_2(\xi, \eta, t_{fin})$

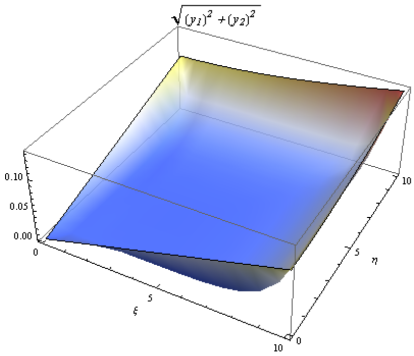

Fig. 6: Final distribution of $|\mathbf{y}|$ *at* $t = t_{fin}$

Density function $z(\xi, \eta, t)$ could be found from Eq. (8). Initial and final distributions of density function are shown in Fig. 7, 8.

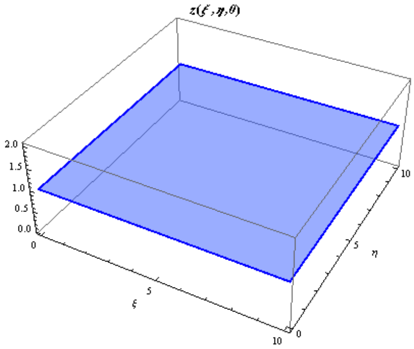

Fig. 7: Initial distribution of $z(\xi, \eta, 0)$

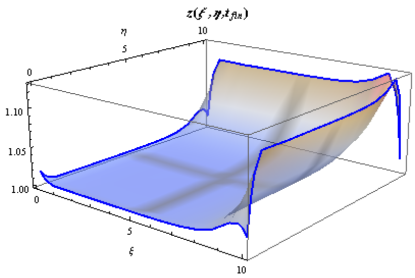

Fig. 8: Final distribution of $z(\xi, \eta, t_{fin})$ *with* $\beta = \sqrt{2}$

Parameter β control the gradient of $z(\xi, \eta, t)$ distribution at time. If it goes to infinity $\beta \to \infty$, then $z(\xi, \eta, t_{fin}) \to z(\xi, \eta, 0)$. For example, if $\beta = 10$ the final distribution of $z(\xi, \eta, t)$ is shown in Fig. 9.

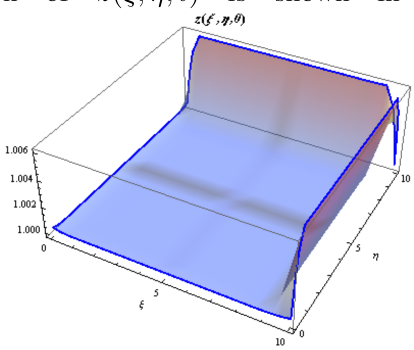

Fig. 9: Final distribution of $z(\xi, \eta, t_{fin})$ *with* $\beta = \sqrt{10}$

Summarize, the final distribution of the charged medium density $z(\xi, \eta, t)$ increase on the border $\xi = 10$, $\eta = 10$ of the square.

The distributions presented in Fig. 8, 9 were computed in a quarter of a square $[-10, 10] \times [-10, 10]$ and the total density function is shown in Fig. 10.

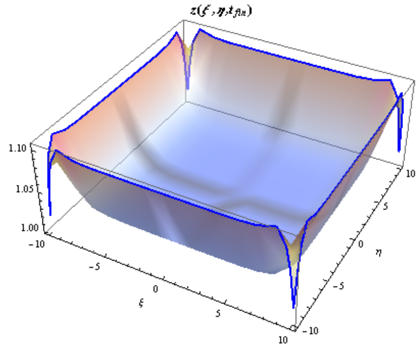

Fig. 10: Total final distribution of $z(\xi, \eta, t_{fin})$ *in square* $[-10, 10] \times [-10, 10]$

4. CONCLUSION

- General system of equations (5)—(7) was reduced to a divergence type equation (14) and system (18), (19).
- Equation (14) was solved numerically in partial **case A** and the final distribution of the charged medium density $z(\xi, \eta, t_{fin})$ was found.
- Such type of the final distribution $z(\xi, \eta, t_{fin})$ is thought to be a part of two dimensional periodic structure.
- Note that all suggestions are performed in three dimensions also.

REFERENCES

[1] Pustovoit V.I., Journal of Comm. Tech. and Electr. (2006), **8**, 937-943.

[2] Moreau, E. and O. Vallee, Phys. Rev. E , Vol. 73, 016112 (4 p.), 2006.

[3] E. Ott, W. M. Manheimer, D. L. Book, and J. P. Boris, Phys. Fluids (1973), **16**, 855-862.

[4] Kravchenko O.V., PIERS Proc. (2012), Moscow, Russia, 1228-1231.